Multicast Transmissions in Non-Cooperative Networks with a Limited Number of Selfish Moves

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Abstract. We study a multicast game in communication networks in which a source sends the same message or service to a set of destinations and the cost of the used links is divided among the receivers according to given cost sharing methods. Assuming a selfish and rational behavior, each receiving user is willing to select a strategy yielding the minimum shared cost. A Nash equilibrium is a solution in which no user can decrease its payment by adopting a different strategy, and the price of anarchy is defined as the worst case ratio between the overall communication cost yielded by an equilibrium and the minimum possible one. Nash equilibria requiring an excessive number of steps to be reached or being hard to compute or not existing at all, we are interested in the determination of the price of anarchy reached in a limited number of rounds, each of which containing at least one move per receiving user. We consider different reasonable cost sharing methods, including the well-known Shapley and egalitarian ones, and investigate their performances versus two possible global criteria: the overall cost of the used links and the maximum shared cost of users. We show that, even in case of two receivers making the best possible move at each step, the number of steps needed to reach a Nash equilibrium can be arbitrarily large. Moreover, we determine the cost sharing methods for which a single round is already sufficient to get a price of anarchy comparable to the one at equilibria, and the ones not satisfying such a property. Finally, we show that finding the sequence of moves leading to the best possible global performance after one-round is already an intractable problem, i.e., NP-hard.

Keywords. Multicast, Nash equilibria, price of anarchy, limited number of best responses.

1 Introduction

Multicast protocols like the IP over the Internet are bandwidth-conserving technologies that reduce traffic over a network by simultaneously delivering the same message or service of a given source station s to a set R of receivers. Applications that take advantage of multicast include videoconferencing, corporate communications, distance learning, distribution of software, stock quotes, and news [8]. Unlike unicast transmissions, where in order to send the same message to multiple receivers a source has to send a copy of this message to each receiver, in multicast transmissions the source sends a single message to all the receivers and when the message reaches a branch point the router duplicates it and then sends a copy over each down-stream link. As the bandwidth used by a transmission is not attributable to a single receiver, a natural arising issue is that of finding a way to distribute the cost among all the receivers in some fashion.

In large-scale scenarios, such as the Internet, there is no authority possible to enforce a centralized traffic management. In such situations, game theory and especially the concepts of Nash equilibria [26] are a suitable framework. If we allow as strategies for each receiver $t \in R$ the set $\mathcal{P}_t$ of the paths from s to t (briefly, (s,t)-paths), a solution is obtained as the outcome of a $|R|$-player game in which receivers t (players) can sequentially modify their strategy by selfishly choosing a different (s,t)-path with the aim of minimizing their shared cost, expressed in terms of a publicly known *cost sharing method*, which specifies how to share the overall cost of the transmission among the receivers belonging to R. Namely, a solution is a *path system* $\mathcal{P}$ containing an (s,t)-paths for every receiver $t \in R$, and the global cost $cost(\mathcal{P})$ to be shared among all the receivers according to a cost sharing method is obtained by summing up the cost of all the links belonging to $\mathcal{P}$. A path system $\mathcal{P}$ is a Nash equilibrium if no player has an incentive to secede in favor of a different solution.

The main algorithmic issues coming from this model include: proving the existence of a Nash equilibrium[1], proving the convergence to a Nash equilibrium from any initial configuration of the players' strategies, estimating the convergence time (i.e. the number of moves necessary to reach an equilibrium starting from an arbitrary configuration), finding Nash equilibria having particular properties (for instance, the one minimizing the global cost or minimize the maximum shared cost), and measuring the *price of anarchy* [22], that is the worst case ratio between the optimal social solution and a Nash equilibrium. Often Nash equilibria may not exist or it may be hard to compute or the time for convergence to Nash equilibria may be extremely long, even if the players always choose a *best response move*, i.e. a move providing them the smallest possible shared cost. Thus, recent research effort [25] concentrated in the evaluation of the speed of convergence (or non-convergence) to an equilibrium in terms of *covering walks*, where a covering walk consists of a sequence of best response moves of the receivers, with each receiver appearing at least once in each walk. As a special case, a *one-round walk* is defined as a covering walk such that each receiver plays exactly one best response move. Moreover, another important issue is evaluating the loss of social performance in selfish evolutions with a (polinomially) bounded number of moves, not necessary terminating in a Nash equilibrium.

Related Work. Several games [12, 13, 16, 24, 30, 33] have been shown to possess pure Nash equilibria or to converge to a pure Nash equilibrium independently from their starting state. An interesting work estimating the convergence time to Nash equilibria is [10] and in [7] finding Nash equilibria having particular properties has been shown to be NP-complete. Considerable research effort has been also devoted to analyze the price of anarchy in different settings, such as in wireless and all-optical networks [3, 5, 21, 23, 31].

The multicast cost sharing problem has been largely investigated both in standard networks [1, 14, 15, 20, 27, 29] and in wireless networks [4, 28], where the cost shared among the receivers is the overall power consumption, also in the *mechanism design* framework.

Mirrokni and Vetta [25] addressed the convergence to approximate solutions in basic-utility and valid-utility games. They proved that starting from any state, one-round of selfish behavior of players converges to a 1/3-approximate solution in basic-utility games. Goemans, Mirrokni and Vetta [18] studied a new equilibrium concept (i.e. sink equilibria) inspired from convergence on best-response walks and proved a fast convergence to approximate solutions on best response walks in (weighted) congestion games. Other related papers studied the convergence for different classes of games such as load balancing games [11], market sharing games [19], and potential and cut games [6].

Our Contribution. In this paper we consider the multicast games induced by four natural cost sharing methods which distribute the cost as follows: (i) in an *egalitarian* way, that is by equally distributing the overall cost among all the receivers; (ii) in a *path-proportional* way,

[1] Indeed, Nash proved that a randomized equilibrium always exists, while we are interested in pure Nash equilibria.

that is by distributing the cost of each link among its down-streaming receivers proportionally to the overall cost their chosen path requires; (iii) in an *egalitarian-path-proportional* way, that is by distributing the overall transmission cost among all the receivers proportionally to the cost of their chosen path; (iv) by applying the definition of the *Shapley Value* [32], that is by equally distributing the cost of each link among all the down-stream receivers.

We first prove that, while the game yielded by the path-proportional cost sharing method in general does not admit a Nash equilibrium, the other three methods yield games always converge to a Nash equilibrium starting from any initial configuration. We then show that the price of anarchy for the game yielded by the egalitarian method is unbounded and provide matching upper and lower bounds for the price of anarchy of the other two convergent games with respect to two different social cost functions, that is the overall transmission cost (function *cost*), which coincides with the sum of all the shared costs, and the maximum shared cost paid by the receivers (function *max*). Unfortunately, for both such metrics the price of anarchy is the worst possible one, that is equal to the number of receivers. Moreover, for the methods inducing convergent games, we prove that even with only two receivers, the number of best responses needed to reach a Nash equilibrium starting from an arbitrary configuration can be arbitrarily high.

Motivated by the previous results, we evaluate the price of the anarchy after a limited number of best responses, for all the proposed methods including the path-proportional one, which may do not admit a Nash equilibrium. We prove that for the egalitarian and path-proportional methods the price of the anarchy is unbounded for any sequence of best response moves and one-round walks, respectively. For the more interesting egalitarian-path-proportional and Shapley value methods we provide tight or almost tight bounds for one-round and covering walks. Such results have been determined for both the two different global cost functions *cost* and *max*.

Finally, we show that finding the best permutation of receivers moves leading to the lowest possible social cost after a one-round walk is already an intractable problem, i.e., NP-hard.

The paper is organized as follows. In the next section we present some basic definitions and notation. In Section 3 we present some preliminaries results concerning the Nash equilibria for the multicast routing problem, and we show that the number of moves necessary to reach a Nash equilibrium can be arbitrarily high. In Section 4 we provide upper and lower bounds on the price of anarchy after a one-round walk or a covering walk, for the social function *cost*. In Section 5 we prove the intractability result and in Section 6 we briefly extend the proofs to the social function *max*. Finally, in Section 7 we give some conclusive remarks and discuss some open questions.

2 Definitions and Notation

A communication network is usually modelled as a graph $G(V, E, c)$ in which $V = \{1, \ldots, n\}$ is a set of intercommunicating nodes, $E \subseteq V \times V$ is a set of m links between the nodes and $c : E \mapsto \mathbb{R}^+$ is a *cost function* associating to each link (t, t') a *transmission cost*, that is the cost for exchanging messages between nodes t and t'. Given a distinguished source node $s \in V$, we identify with $R \subseteq V - \{s\}$ the set of all the nodes interested in receiving the transmission from the source s. Let us denote by $c(p_t) = \sum_{e \in p_t} c(e)$ the overall transmission cost of a path p_t.

Given a path system $\mathcal{P}$, the global cost to be shared among all the receivers is obtained by summing up the cost of all the links belonging to $\mathcal{P}$, i.e., $cost(\mathcal{P}) = \sum_{e \in E'} c(e)$, where $E' = \bigcup_{p \in \mathcal{P}} \bigcup_{e \in p} \{e\}$.

A *cost sharing method* is a function $\mathcal{M}$ which, given a set of receivers R with an associated path system $\mathcal{P}$, distributes among all the receivers the total cost $cost(\mathcal{P})$ associated with $\mathcal{P}$

in such a way that $\sum_{t \in R} \mathcal{M}(\mathcal{P}, t) = cost(\mathcal{P})$, where $\mathcal{M}(\mathcal{P}, t)$ is the cost attributed to the receiver t.

We consider the following four natural cost sharing methods:

- $\mathcal{M}_1$ (*egalitarian* [9]) distributes the cost by equally sharing the global cost among all the receivers, i.e., $\mathcal{M}_1(\mathcal{P}, t) = \frac{cost(\mathcal{P})}{|R|}$.
- $\mathcal{M}_2$ (*path-proportional*) distributes the cost of each link $e \in E'$ among all the downstreaming receivers t' using e proportionally to the overall cost their chosen (s, t')-path requires, i.e., $\mathcal{M}_2(\mathcal{P}, t) = \sum_{e \in p_t} c(e) \frac{c(p_t)}{\sum_{t': e \in p_{t'}} c(p_{t'})}$.
- $\mathcal{M}_3$ (*egalitarian-path-proportional*) distributes the overall cost among all the receivers proportionally to the cost of their chosen path, i.e., $\mathcal{M}_3(\mathcal{P}, t) = cost(\mathcal{P}) \frac{c(p_t)}{\sum_{t' \in R} c(p_{t'})}$
- $\mathcal{M}_4$ (*Shapley* [32]) equally distributes the cost of each link among all the receivers using it, i.e., $\mathcal{M}_4(\mathcal{P}, t) = \sum_{e \in p_t} \frac{c(e)}{l(\mathcal{P}, e)}$ where $l(\mathcal{P}, e) = |\{t \in R \mid e \in p_t,\, p_t \in \mathcal{P}\}|$ is the number of receivers using link e for their transmission.

A *Nash equilibrium* for $\mathcal{G}$ is a path system $\mathcal{P}$ such that $\forall t \in R$ and path $p'_t \in \mathcal{P}_t$ inducing a new path system $\mathcal{P}' = \mathcal{P} \setminus \{p_t\} \cup \{p'_t\}$, it holds $\mathcal{M}(\mathcal{P}, t) \leq \mathcal{M}(\mathcal{P}', t)$. Denoting with $\mathcal{N}$ the set of all the possible Nash equilibria for the game $\mathcal{G}$, the *price of anarchy* is defined as the worst case ratio among the Nash versus optimal performance, that is $\rho(\mathcal{G}) = \max_{\mathcal{P} \in \mathcal{N}} \frac{cost(\mathcal{P})}{cost(\mathcal{P}^*)}$ where $\mathcal{P}^*$ is a path system of minimum cost for the multicast routing.

In order to model the selfish behavior of the receivers, let us introduce the notion of *state graph*.

Definition 1. *A* state graph *is a directed graph having a node for any possible path system $\mathcal{P}$ and an arc $(\mathcal{P}, \mathcal{P}')$ with label t if $\mathcal{P}$ and $\mathcal{P}'$ differ only for the choice of t and both these conditions are met: (i) $\mathcal{M}(\mathcal{P}', t) \leq \mathcal{M}(\mathcal{P} - \{p_t\} \cup \{p'_t\}, t)$ for any $p'_t \in \mathcal{P}_t$; (ii) if $\mathcal{P} \neq \mathcal{P}'$, $\mathcal{M}(\mathcal{P}', t) < \mathcal{M}(\mathcal{P}, t)$.*

Notice that the graph may contain loops, and there is an arc $(\mathcal{P}, \mathcal{P}')$ labelled t if and only if t, starting from $\mathcal{P}$, can play a best response move such that the resulting path system is $\mathcal{P}'$.

Given a best response walk starting from an arbitrary state, we are interested in the social value of the last state of the walk. Notice that if we do not allow every player to make a best response on a walk P, then we cannot bound the social value of the final state with respect to the optimal solution. This follows from the fact that the actions of a single player may be very important for producing solutions of high social value. Motivated by this simple observation, Mirrokni and Vetta [25] introduced the following models that capture the intuitive notion of a fair sequence of moves:

One-round walk. Consider an arbitrary ordering of all receivers $i_1, \ldots, i_{|R|}$. A walk of length $|R|$ in the state graph is a one-round walk if its arcs are labelled $i_1, \ldots, i_{|R|}$ in this order.

Covering walk. A walk in the state graph is a covering walk if for each player i, it has at least one arc with label i.

k-Covering walk. A walk in the state graph is a k-covering walk if it can be split in k disjoint covering walks.

Note that unless otherwise stated, all walks are assumed to start from an arbitrary initial state.

3 Existence and convergence to Nash equilibria

As an extension of our work on Nash equilibria in multicast transmissions in wireless networks [2], it is not difficult to prove the following results on the existence of Nash equilibria and to provide matching upper and lower bounds for the price of anarchy. Details will appear in the full version of the paper.

Theorem 1. *The games $\mathcal{G} = (G, R, \mathcal{M}_i)$, $i \in \{1, 3, 4\}$, always converge to a Nash equilibrium, for any network G and receivers' set R. On the other hand, the game $\mathcal{G} = (G, R, \mathcal{M}_2)$ may not have a Nash equilibrium.*

Theorem 2. *The price of anarchy of the multicast transmission game $\mathcal{G} = (G, R, \mathcal{M}_i)$ is unbounded for $i = 1$ and is $|R|$ for $i \in \{3, 4\}$.*

For the cost sharing methods inducing games which always admit a Nash equilibrium, the number of best response moves necessary to reach an equilibrium starting from an arbitrary configuration is unbounded even for a game with 2 receivers. In fact, the following theorem holds.

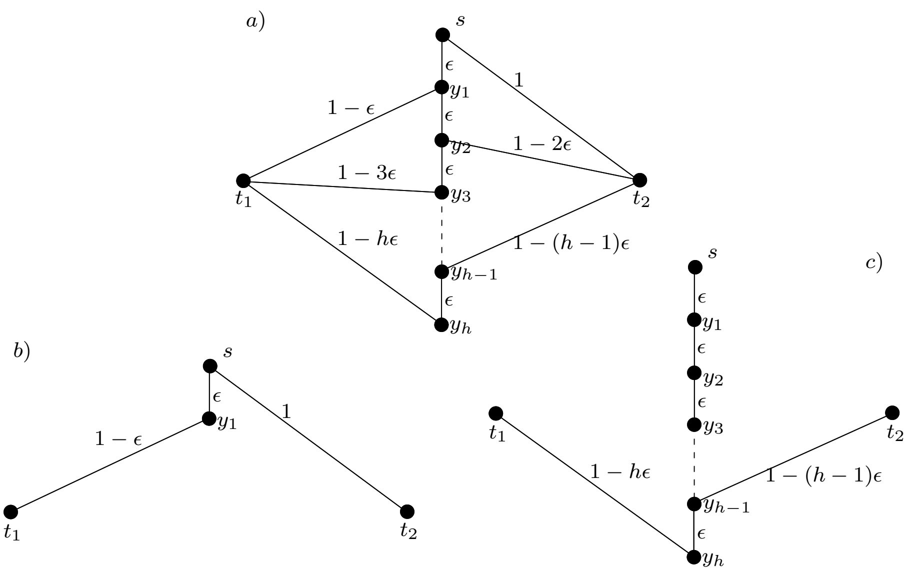

Fig. 1. a) The communication network. b) The initial configuration. c) The Nash equilibrium.

Theorem 3. *Given any integer $h > 0$, there exist a network G and an initial state starting from which the number of best response moves necessary to reach a Nash equilibrium for the game $\mathcal{G} = (G, R, \mathcal{M}_i)$, $i \in \{1, 3, 4\}$, is greater than h even if $|R| = 2$.*

Proof. Consider the communication network depicted in Figure 1a, where $R = \{t_1, t_2\}$ and $\epsilon > 0$ is such that $h\epsilon < \frac{1}{2}$. Consider the evolution of the game induced by the cost sharing method $\mathcal{M}_i$ for $i \in \{1, 2, 3\}$, starting from the state corresponding to the path system depicted in Figure 1b. Moreover assume that t_1 and t_2 move alternately, starting from t_1, and each receiver always chooses the path with the smallest number of links among those with the smallest cost.

Since $1 - h\epsilon > \frac{1}{2}$, no receiver can choose as its best move a path containing more than one edge of cost $1 - i\epsilon, i = 0, 1, \ldots, h$. It's easy to see that the number of moves executed in order to reach the equilibrium is h. More precisely, considering the ordered list of moves $j = 1, 2, \ldots, h$, the odd moves are executed by t_1 and the even ones by t_2; each receiver chooses the path containing the edge of cost $1 - j\epsilon$ and j edges of cost ϵ. □

4 One-round and covering walks

In this section, we analyze the price of anarchy after a limited number of best response moves. More precisely, we provide lower bounds for one-round walks and upper bounds for covering walks; since a one-round walk is a special case of a covering one, the lower and upper bounds also hold for covering and one-round walks, respectively.

For the cost sharing methods $\mathcal{M}_1$ and $\mathcal{M}_2$, it is possible to show that the price of anarchy after one-round walks is unbounded.

Theorem 4. *Given any integer h, there exists an instance of the game $\mathcal{G} = (G, R, \mathcal{M}_1)$ for which the price of anarchy after a one-round walk is at least h.*

Proof. The claim is a direct consequence of Theorem 2. Since for any h there exists a Nash equilibrium of cost h times the optimal one, starting from it every best response walk cannot leave the initial configuration, thus achieving a price of anarchy equal to h after a one-round walk. □

Notice that this result holds for any sequence of best response moves, including k-covering paths for any $k \geq 1$.

Concerning the cost sharing method $\mathcal{M}_2$, as shown in Theorem 1, it may not admit a Nash equilibrium. However, again it is interesting to estimate the price of anarchy after a limited number of best response moves. Unfortunately, also in this case the price of anarchy after a one-round walk is unbounded.

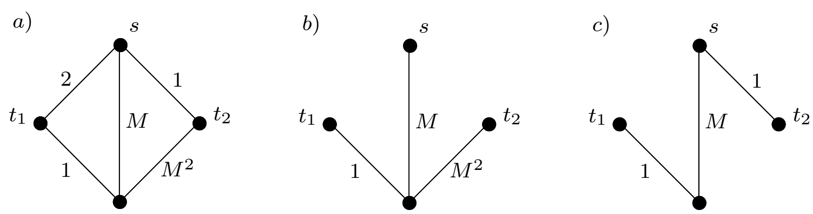

Fig. 2. a) The communication network. b) The initial configuration. c) The final configuration.

Theorem 5. *Given any integer h, there exists an instance of the game $\mathcal{G} = (G, R, \mathcal{M}_2)$ for which the price of anarchy after a one-round walk is at least h.*

Proof. Consider the communication network depicted in Figure 2a, where $R = \{t_1, t_2\}$. Consider the one-round walk in which t_1 moves before t_2, starting from the state that corresponds to the path system depicted in Figure 2b. Since receiver t_1 in the initial configuration is paying $M \frac{M}{M^2+2M} < 1$, its best move consists in remaining on its initial path, while t_2 later moves on the path consisting of a unique edge of cost 1. The overall cost of the path system associated to the final configuration, depicted in Figure 2c, is $M + 2$. On the other side, the optimal solution costs exactly 3. Thus, the claim directly follows by choosing $M > 3h - 2$. □

The remaining cost sharing methods $\mathcal{M}_4$ and $\mathcal{M}_3$ are more interesting since it is possible to bound the price of anarchy after a one-round or a covering walk in a tight or almost tight way, respectively.

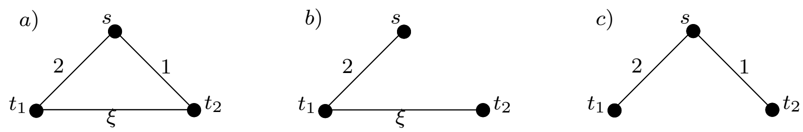

Fig. 3. a) The communication network. b) The initial configuration. c) The final configuration.

4.1 Egalitarian path proportional cost sharing method $\mathcal{M}_3$

For the cost sharing method $\mathcal{M}_3$ we provide matching upper and lower bounds for $|R| = 2$, and almost matching lower and upper bounds for $|R| > 2$.

Theorem 6. *Given any $\epsilon > 0$, there exists an instance of the game $\mathcal{G} = (G, R, \mathcal{M}_3)$ where $|R| = 2$ for which the price of anarchy after a one-round walk is at least $3 - \epsilon$.*

Proof. Consider the communication network depicted in Figure 3a, where $R = \{t_1, t_2\}$ and $\xi > 0$. Consider a one-round walk in which t_1 moves before t_2, starting from the state that corresponds to the path system depicted in Figure 3b. It is easy to check that the best move for t_1 is to remain on its initial path, since $\frac{2}{4+\xi}(2+\xi) \leq \frac{1+\xi}{3+2\xi}(3+\xi)$. Thus, the overall cost of the path system we obtain after a one-round walk, depicted in Figure 3c, is 3, while the optimal solution costs exactly $1 + \xi$. Since the price of the anarchy after a one-round walk is $\frac{3}{1+\xi}$, the claim follows by choosing $\xi < \frac{\epsilon}{3-\epsilon}$. □

Theorem 7. *The price of anarchy after a covering walk for the game $\mathcal{G} = (G, R, \mathcal{M}_3)$ where $|R| = 2$ is at most 3.*

Proof. We refer to the *last move* of a receiver as its last move in the covering walk. Let t_1 be the receiver doing its last move before t_2. Moreover, for $i \in \{1, 2\}$, let m_i and p_i be a shortest path connecting s to t_i and the s-t_i path chosen by t_i at its last move, respectively.

Since $cost(\mathcal{P}) \geq \frac{\sum_{t' \in R} c(p_{t'})}{2}$, recalling the definition of the cost sharing method $\mathcal{M}_3$, we obtain $c(p_1) \leq 2c(m_1)$, because otherwise m_1 would be a better path for t_1. Moreover, since $cost(\mathcal{P}) \leq \sum_{t' \in R} c(p_{t'})$, a player never pays for a path p more than $c(p)$. Therefore, t_1 pays at the end of the covering walk at most $2c(m_1)$.

Using similar arguments, and considering that no receiver moves after t_2, it is possible to show that t_2 pays at the end of the covering walk at most $c(m_2)$.

Since an optimal routing has cost at least equal to $max\{c(m_1), c(m_2)\}$, the price of anarchy after a covering walk is at most

$$\frac{2c(m_1) + c(m_2)}{max\{c(m_1), c(m_2)\}} \leq \frac{2max\{c(m_1), c(m_2)\} + max\{c(m_1), c(m_2)\}}{max\{c(m_1), c(m_2)\}} = 3.$$

□

Starting from the results of Theorem 2 and using arguments similar to the ones exploited in the proof of Theorem 4, it is not difficult to prove the following theorem.

Theorem 8. *There exists an instance of the game $\mathcal{G} = (G, R, \mathcal{M}_3)$ for which the price of anarchy after a one-round walk is at least $|R|$.*

In order to derive a suitable upper bound, in the following lemma we first show an interesting property correlating the cost of the path chosen by a receiver t doing a best response move with the one of the shortest path connecting the source with t.

Lemma 1. *Let p_t be the path chosen by player t via a best response move, and m_t be the minimum cost path connecting s to t. Then, $c(p_t) \leq (1 + \phi)c(m_t)$, where $\phi \approx 1,618$ is the golden number.*

Proof. Let $A = \sum_{t' \in R - \{t\}} c(p_{t'})$ be the sum of the cost of the path used by the other receivers and B be the overall cost of the path system discarding t, just before the best response move of t. Moreover, let $x = c(p_t)$ and $m = c(m_t)$. Clearly, $x \geq m$. Since t plays a best response move, the following *base inequality* holds:

$$\frac{x}{A+x}(B+\delta_x) \leq \frac{m}{A+m}(B+\delta_m),$$

where δ_x and δ_m are the costs of the edges in x and m, respectively, not contained in the paths used by the other receivers.
We distinguish two different cases.

- $\delta_x \geq \frac{x}{1+\phi}$. Since by the base inequality it must hold that $B + \delta_x \leq B + \delta_m$, we obtain $B + \frac{x}{1+\phi} \leq B + \delta_x \leq B + \delta_m \leq B + m$. Thus, $x \leq (1+\phi)m$.
- $\delta_x < \frac{x}{1+\phi}$. We have that $B \geq x - \frac{x}{1+\phi}$.
  The proof is again divided into two disjoint subcases.
  - If $A \cdot B \leq m^2$, since $A \geq B$, it follows that $B \leq m$. Thus, $x \leq \frac{1+\phi}{\phi} m < (1+\phi)m$.
  - If $A \cdot B > m^2$, by the base inequality we obtain that $x \leq \frac{AB+Am}{AB-m^2} m = \frac{B+m}{B-\frac{m^2}{A}} m \leq \frac{B^2+Bm}{B^2-m^2} m$.
    Moreover, since $x \leq \phi B$, we have that $\frac{x}{m} \leq \min\{\frac{\phi B}{m}, \frac{B^2+Bm}{B^2-m^2}\}$.
    Since, fixed B, $\frac{\phi B}{m}$ is a decreasing function in m and $\frac{B^2+Bm}{B^2-m^2}$ is increasing in m, the minimum is maximized in the intersection of the two functions, which is obtained for $m = \frac{B}{\phi}$, where both of them assume value $1+\phi$. □

We are now ready to prove the following theorem, that provides an almost matching upper bound for the price of anarchy after a covering walk.

Theorem 9. *The price of anarchy after a covering walk for the game $\mathcal{G} = (G, R, \mathcal{M}_3)$ is at most $(1+\phi)(|R|-1)+1$, where $\phi \approx 1,618$ is the golden number.*

Proof. For $i \in \{1, 2, \ldots, |R|\}$, let m_i be a shortest path connecting s to r_i.

Recalling the definition of the cost sharing method $\mathcal{M}_3$, it is easy to see that $cost(\mathcal{P}) \leq \sum_{t' \in R} c(p_{t'})$. Thus, a player never pays for a path p more than $c(p)$. By Lemma 1, each player at the end of a covering walk pays at most $(1+\phi)c(m_t)$.

Moreover, the last player of the walk, say receiver $t_{|R|}$, at the end of the covering walk pays at most $c(m_{|R|})$, because otherwise $m_{|R|}$ would be a better path.

Since an optimal routing has cost at least equal to $max\{c(m_1), \ldots, c(m_{|R|})\}$, the price of anarchy after a covering walk is at most

$$\frac{(1+\phi)c(m_1) + \ldots + (1+\phi)c(m_{|R|-1}) + c(m_{|R|})}{max\{c(m_1), \ldots, c(m_{|R|})\}} \leq (1+\phi)(|R|-1)+1.$$

□

4.2 Shapley cost sharing method $\mathcal{M}_4$

For the cost sharing method $\mathcal{M}_4$ we provide matching upper and lower bounds. Unfortunately, the price of anarchy after a covering walk is very high, i.e. its order is quadratic in the number of receivers.

Theorem 10. *Given any $\epsilon > 0$, there exists an instance of the game $\mathcal{G} = (G, R, \mathcal{M}_4)$ for which the price of anarchy after a one-round walk is at least $\frac{|R|(|R|+1)}{2} - \epsilon$.*

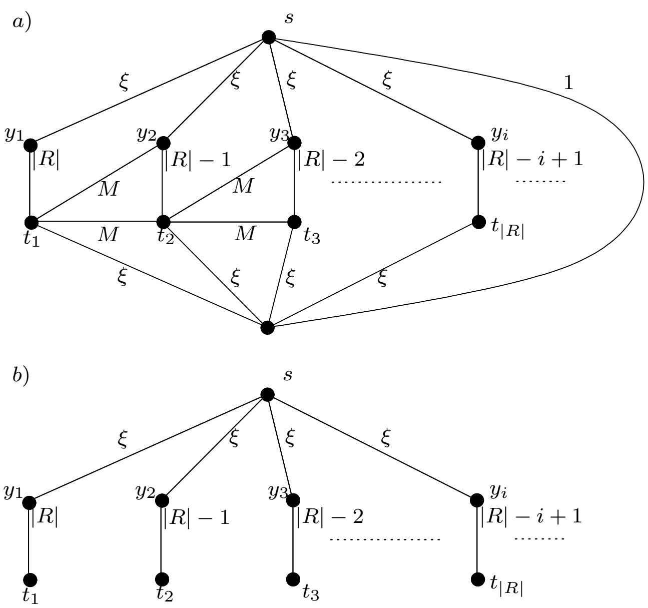

Fig. 4. a) The communication network. b) The final configuration.

Proof. Consider the communication network depicted in Figure 4a, where $R = \{t_1, t_2, \ldots, t_{|R|}\}$ and $M = |R|^2$. Let $t_1, \ldots, t_{|R|}$ be the sequence of the receivers such that $i < j$ if and only if the move of t_i precedes the one of t_j in a one-round walk starting from the state that corresponds to the path system in which each receiver t_i, $i = 1, \ldots, |R|$, uses the path $\{\{t_i, t_{i-1}\}, \{t_{i-1}, y_{i-1}\}, \{y_{i-1}, t_{i-2}\}, \ldots, \{t_1, y_1\}, \{y_1, s\}\}$.

It is easy to check that each receiver t_i, $i = 1, \ldots, |R|$, choose as its best move the two edges path $\{\{t_i, y_i\}, \{y_i, s\}\}$. Thus, after a one-round walk we obtain a path system (depicted in Figure 4b) of cost $(|R| + \xi) + (|R| - 1 + \xi) + \ldots + (1 + \xi) = \frac{|R|(|R|+1)}{2} + |R|\xi$. Since the optimal solution costs exactly $1 + |R|\xi$, the claim follows by choosing $\xi < \frac{\epsilon}{|R|^3}$.

□

Theorem 11. *The price of anarchy after a covering walk for the game $\mathcal{G} = (G, R, \mathcal{M}_4)$ is at most $\frac{|R|(|R|+1)}{2}$.*

Proof. We refer to the *last move* of a receiver as its last move in the covering walk. Let $t_1, \ldots, t_{|R|}$ be the sequence of the receivers such that $i < j$ if and only if the last move of t_i precedes the last move of t_j in the covering walk. Moreover, for $i \in \{1, \ldots, |R|\}$, let m_i, x_i and p_i be a shortest path connecting s to t_i, the cost payed by player t_i just after its last move and the s-t_i path chosen by t_i at its last move, respectively.

Consider the last move of receiver t_i. Clearly, $x_i \leq c(m_i)$, otherwise m_i would be a better path for t_i. For each edge e of p_i, let $x_{i,e}$ and $y_{i,e}$ be the payment of t_i for edge e just after its last move and at the end of the covering walk, respectively, and let $h_{i,e} = h'_{i,e} + h''_{i,e}$ be the number of receivers sharing edge e with t_i just after its last move, where $h'_{i,e}$ is the number of receivers t_j with $j < i$, i.e. not moving after the last move of t_i.

The payment of t_i at the end of the covering walk is

$$\sum_{e \in p_i} y_{i,e} \leq \sum_{e \in p_i} x_{i,e} \frac{1 + h'_{i,e} + h''_{i,e}}{1 + h'_{i,e}} \leq \sum_{e \in p_i} x_{i,e}(1 + h''_{i,e}) \leq$$

$$\leq (|R| - i + 1) \sum_{e \in p_i} x_{i,e} \leq (|R| - i + 1)c(m_i).$$

Since an optimal routing has cost at least equal to $max\{c(m_1), \ldots, c(m_{|R|})\}$, the price of anarchy after a covering walk is at most

$$\frac{|R|c(m_1) + (|R|-1)c(m_2) + \ldots + c(m_{|R|})}{max\{c(m_1), \ldots, c(m_{|R|})\}} \leq |R| + (|R|-1) + \ldots + 1 = \frac{|R|(|R|+1)}{2}.$$

□

5 Computing the best one-round evolution is NP-hard

A network provider could be interested in determining a proper permutation of the receivers such that, letting the receivers move in the specified order, the final configuration after a one-round walk is ensured to have the lowest possible social cost.

Under this scenario, in this section we prove that determining such a permutation for the games $\mathcal{G} = (G, R, \mathcal{M}_i)$ for $i \in \{1, 2, 3, 4\}$ is computationally hard.

Theorem 12. *Consider the cost sharing methods $\mathcal{M}_i$, $i \in \{1, 2, 3, 4\}$. Computing the permutation of receivers such that, letting the receivers move in the specified order, the final configuration after the one-round walk is ensured to have the lowest possible social cost is an NP-hard problem.*

Due to space limitations, the proof is in Appendix.

Finally, by using the same reduction, since the best response moves are always unique, also the problem of determining the best one-round, i.e. the permutation and the best response move for each agent, leading to the best social value, is NP-Hard.

6 The maximum shared cost social function

In this section we show how to extend our results to the case in which the considered social function *max* is given by the maximum shared cost of the receivers, that is $max(\mathcal{P}, \mathcal{M}_i) = max_{t \in R}\mathcal{M}_i(\mathcal{P}, t)$, for $i \in \{1, 2, 3, 4\}$.

Due to space limitation, we just restrict in outlining the basic differences with respect to the previous analyzed social function given by the overall cost of the used path system. More details will appear in the full version of the paper.

Concerning the price of anarchy of Nash equilibria, it remains unbounded for the cost sharing method $\mathcal{M}_1$, and is again equal to $|R|$ for the cost sharing methods $\mathcal{M}_3$ and $\mathcal{M}_4$.

For the cost sharing methods $\mathcal{M}_1$ and $\mathcal{M}_2$, the price of anarchy remains unbounded for k-covering walks, with any k, and one-round walks, respectively.

For the cost sharing method $\mathcal{M}_3$, in networks with 2 receivers we can derive a lower bound equal to 2 for a one-round walk, and an exactly matching upper bound for a covering walk. Moreover, for more than 2 receivers we have a lower bound equal to R for a one-round walk, and an upper bound of $(1 + \phi)|R|$ for a covering walk, where $\phi \approx 1,618$ is the golden number.

For the cost sharing method $\mathcal{M}_4$, we can derive matching lower and upper bounds equal to $|R|^2$ for a one-round walk and a covering walk, respectively.

Finally, the problem of determining a proper permutation of the receivers such that, letting the receivers move in the specified order, the final configuration after a one-round walk is ensured to have the lowest possible social cost, remains NP-hard in the case of the social function *max*, for all the four considered cost sharing methods.

7 Conclusions

We have investigated the price of anarchy of the selfish game arising by multicasting in non-cooperative networks with respect to four basic cost sharing methods and two different social functions.

Many question are left open.

First of all, for the Shapley cost sharing method $\mathcal{M}_4$, it would be nice to determine the minimum number of rounds sufficient to guarantee a price of anarchy proportional to the one at equilibrium. In fact, while $\mathcal{M}_3$ satisfies this property and for $\mathcal{M}_1$ the price of anarchy can be unbounded after any number of rounds, $\mathcal{M}_4$ is the only cost sharing method that after one-round has a price of anarchy quadratic with respect to the one at equilibrium. Even if we do not have a formal proof yet, for $\mathcal{M}_4$ we conjecture that two rounds are already enough.

On this respect, an exception holds for $\mathcal{M}_2$, for which we have proven that the price of anarchy is unbounded after one-round, but there may not exist an equilibrium. In this case, it is important to understand what is the best achievable price of anarchy and when it becomes proportional to the number of receivers.

It would also be nice to refine our results by determining when possible the number of rounds needed to exactly reach the price of anarchy at equilibrium.

Finally, a worth investigating issue is that of considering also the price of stability, that is the best possible performance achievable at equilibrium or after a fixed number of moves.

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Appendix

Proof of Theorem 12

We prove the claim by exploiting a reduction from the *Exact 3-Cover* problem well known to be NP-hard [17]. In this problem we are given a universe $X = \{x_1, \ldots, x_{3n}\}$ of $3n$ elements and a collection $\mathcal{C} = \{C_1 \ldots, C_m\}$ of m subsets of X such that $|C_j| = 3$ for any $1 \leq j \leq m$ and $\bigcup_{j=1}^{m} C_j = X$. The objective is to find a collection of subsets $\mathcal{F} \subseteq \mathcal{C}$ such that $\mathcal{F} = \{C_{i_1}, \ldots, C_{i_n}\}$ and $\bigcup_{j=1}^{n} C_{i_j} = X$.

Given an instance $(X, \mathcal{C})$ of Exact 3-Cover, we can construct an instance of the multicast transmission games $\mathcal{G} = (G, R, \mathcal{M}_i)$, $i \in \{1, 2, 3, 4\}$, i.e. a network G, a receiver set R and an initial configuration, in the following way.

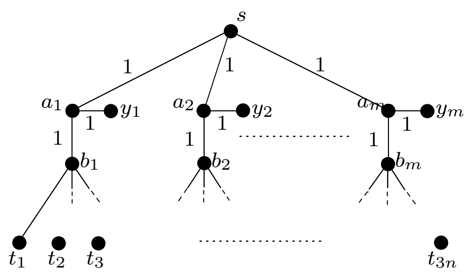

Fig. 5. The communication network corresponding to the instance of Exact 3-Cover

The network G has vertices set $V = \{s\} \cup V_1 \cup V_2 \cup V_3$ and edge set $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$, where

$V_1 = \{a_j, b_j \mid j = 1, \ldots, m\}$,
$V_2 = \{y_j \mid j = 1, \ldots, m\}$,
$V_3 = \{t_i \mid i = 1, \ldots, 3n\}$,

and

$E_1 = \{\{s, a_j\} \mid j = 1, \ldots, m\}$,
$E_2 = \{\{y_j, a_j\} \mid j = 1, \ldots, m\}$,
$E_3 = \{\{a_j, b_j\} \mid j = 1, \ldots, m\}$,
$E_4 = \{\{b_j, t_i\} \mid x_i \in C_j\}$,
$E_5 = \{\{s, y_j\} \mid j = 1, \ldots, m\} \cup \{\{s, x_i\} \mid i = 1, \ldots, 3n\}$.

The network is depicted in Figure 5, where for the sake of clearness the edges belonging to E_5 are not depicted.

The weight function on the edges is defined as follows: $c(e) = 1$ if $e \in E_1 \cup E_2 \cup E_3$, $c(e) = A$ if $e \in E_4$ and $c(e) = B$ if $e \in E_5$, where $B \gg A \gg 1$ are suitable integers. The receiver set R is $V_2 \cup V_3$ and in the initial configuration each receiver uses the path consisting of the unique edge in E_5 connecting it to s. Notice that B can be chosen big enough so as to force each player to change his strategy during the one walk evolution.

If there exists an exact 3-cover for $(X, \mathcal{C})$, than there exists a permutation of the receivers inducing a one-round walk with a final configuration of global cost $3nA + n + 2m$, i.e. using exactly n edges belonging to E_3. In fact, let $C_{h_1^*}, \ldots, C_{h_n^*}$ be the exact 3-cover; the permutation sequentially selects receiver $y_{h_l^*}$ directly followed by the receivers corresponding to the elements in $C_{h_l^*}$, for each $l = 1, \ldots, n$; finally all the receivers corresponding to sets not belonging to the exact cover are selected in any order. First of all, notice that by a proper choice of A, no path containing two edges in E_4 can be a best move. Moreover, the receiver $y_{h_l^*}$ has as unique best move the two edges path $\{\{y_{h_l^*}, a_{h_l^*}\}, \{s, a_{h_l^*}\}\}$, and the receivers associated to the elements belonging to $C_{h_l^*}$ have as unique best move the path of three edges containing the edge $\{s, a_{h_l^*}\}$.

Conversely, if no exact 3-cover exists, in order to reach all the receivers in V_3 at least $n+1$ edges in E_3 must be used, so that no multicast routing can cost equal to or less than $3nA + n + 2m$.

Thus, a permutation inducing a one-round walk of final cost at most $3nA+n+2m$ exists if and only if there exists an exact 3-cover for X.

Notice that the above arguments hold regardless of the particular cost sharing methods among $\mathcal{M}_1, \ldots, \mathcal{M}_4$ used, thus proving the claim.

□