

On the Convergence of Multicast Games in Directed Networks

Angelo Fanelli, Michele Flammini, and Luca Moscardelli

Dept. of Computer Science, University of L'Aquila

Via Vetoio, Loc. Coppito, 67100 L'Aquila, Italy

angelo.fanelli@di.univaq.it, flammini@di.univaq.it, moscardelli@di.univaq.it

ABSTRACT

We investigate the convergence of the price of anarchy after a limited number of moves in the classical multicast communication game when the underlying communication networks is directed. Namely, a subset of nodes of the network are interested in receiving the transmission from a given source node and can share the cost of the used links according to fixed cost sharing methods. At each step, a single receiver is allowed to modify its communication strategy, that is to select a communication path from the source, and assuming a selfish or rational behavior, it will make a best response move, that is it will select a solution yielding the minimum possible payment or shared cost. We determine lower and upper bounds on the price of anarchy, that is the highest possible ratio among the overall cost of the links used by the receivers and the minimum possible cost realizing the required communications, after a limited number of moves under the fundamental Shapley cost sharing method. In particular, assuming that the initial set of connecting paths can be arbitrary, we show an $O(r\sqrt{r})$ upper bound on the price of anarchy after 2 rounds, during each of which all the receivers move exactly once, and a matching lower bound, that we also extend to $\Omega(r^{\frac{1}{k}})$ for any number $k \geq 2$ rounds, where r is the number of receivers. Similarly, exactly matching upper and lower bounds equal to r are determined for any number of rounds when starting from the empty state in which no path has been selected. Analogous results are obtained also with respect to other three natural cost sharing methods considered in the literature, that is the egalitarian, path-proportional and egalitarian-path proportional ones. Most results are also extended to the undirected case in which the communication links are bidirectional.

Categories and Subject Descriptors

F.0 [Theory of Computation]: General

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee.

SPAA '07, June 9–11, 2007, San Diego, California, USA.

Copyright 2007 ACM 978-1-59593-667-7/07/0006 ...\$5.00.

General Terms

Algorithms, Economics, Theory

Keywords

multicast games, limited number of best-response moves, price of anarchy

1. INTRODUCTION

Multicast primitives in interconnection networks achieve a significant traffic reduction with respect to standard routing protocols, as the source sends the same message to all the receivers and this is replicated only at the branch points, where a copy is sent over each down-stream link. Applications that take advantage of such a feature include the IP protocol, videoconferencing, corporate communications, distance learning, distribution of software, stock quotes, and news [9]. As the bandwidth used by a transmission is not attributable to a single receiver, a natural arising issue is that of finding a way to distribute the cost among all the receivers in some fashion. However, in large-scale scenarios, such as the Internet, there is no authority able to enforce a centralized traffic management. In such situations, game theory and especially the concepts of Nash equilibria [28] are a suitable framework. If we allow as strategies for each receiver $t \in R$ the set \mathcal{P}_t of the paths from s to t (briefly, (s, t) -paths), a multicast solution is obtained as the outcome of a $|R|$ -player game in which receivers (players) can sequentially modify their strategy by selfishly choosing a different (s, t) -path with the aim of minimizing their payment. Such a payment is expressed in terms of a publicly known *cost sharing method*, which specifies how to share the overall cost of the transmission among the receivers belonging to R . Namely, a solution is a *path system* \mathcal{P} containing an (s, t) -path for every receiver $t \in R$, and the global cost $cost(\mathcal{P})$ to be shared among all the receivers according to a cost sharing method is obtained by summing up the cost of all the links belonging to \mathcal{P} . A path system \mathcal{P} is a Nash equilibrium if no player has an incentive to secede in favor of a different solution.

The main algorithmic issues coming from this model include: proving the existence of a Nash equilibrium¹, proving the convergence to a Nash equilibrium from any initial configuration of the players' strategies, estimating the convergence time (i.e. the number of moves necessary to reach an equilibrium starting from an arbitrary configuration),

¹Indeed, Nash proved that a randomized equilibrium always exists, while we are interested in pure Nash equilibria.

finding Nash equilibria having particular properties (for instance, the one minimizing the global cost or the maximum shared cost), and measuring the *price of anarchy* [23] and *price of stability* [1], corresponding respectively to the worst and the best case cost ratios between a Nash equilibrium and the optimal social solution.

Several games [3, 4, 12, 13, 17, 24, 25, 31, 32, 34] have been shown to possess pure Nash equilibria or to converge to a pure Nash equilibrium independently from their starting state, and their price of anarchy or stability has been evaluated. An interesting work estimating the convergence time to Nash equilibria is [11] and in [8] finding Nash equilibria having particular properties has been shown to be NP-complete.

Often Nash equilibria may not exist or it may be hard to compute or the time for convergence to Nash equilibria may be extremely long, even if the players always choose a *best response move*, i.e. a move providing them the smallest possible shared cost. Thus, recent research effort [26] concentrated in the evaluation of the speed of convergence (or non-convergence) to an equilibrium in terms of *covering walks*, where a covering walk consists of a sequence of best response moves of the receivers, with each receiver appearing at least once in each walk. As a special case, a *1-round walk* is defined as a covering walk such that each receiver plays exactly one best response move and a *k-round walk* with $k \geq 1$ as the concatenation of k 1-round walks. An important issue raised by the authors is then that of evaluating the loss of social performance in selfish evolutions with a (polynomially) bounded number of walks, not necessarily terminating in a Nash equilibrium.

More precisely, Mirrokni and Vetta [26] addressed the convergence to approximate solutions in basic-utility and valid-utility games. They proved that starting from any state, one-round of selfish behavior of players converges to a 1/3-approximate solution in basic-utility games and 1/2r-approximate solution in valid-utility games, where r is the number of players. Goemans, Mirrokni and Vetta [18] studied a new equilibrium concept (i.e. sink equilibria) inspired from convergence on best-response walks and proved a fast convergence to approximate solutions on best response walks in (weighted) congestion games. Other related papers studied the convergence for different classes of games such as load balancing games [11], market sharing games [19], and potential and cut games [7].

Related Work.

The multicast cost sharing problem has been largely investigated in the literature [2, 15, 16, 22, 29, 30]. Recent papers considered a model of game-theoretical network design [1, 5, 6, 14] initially proposed and studied by Anshelevich et al. [1], where selfish players select paths in a network so as to minimize their payment, which is prescribed by Shapley cost shares [33]. Namely, if all the players are identical, the cost share incurred by a player for a link in its path is the fixed cost of the link divided by the number of players using it. As remarked in [1], the resulting game belongs to the widely investigated class of congestion games [20, 25, 27, 31, 35], first defined by Rosenthal [31], that the author by means of potential function arguments showed to always possess pure Nash equilibria.

Anshelevich et al. [1] also proved that the price stability is $O(\log r)$, where r is the number of players, both for directed

and undirected networks, and provided a matching lower bound for directed networks. For undirected networks no non-trivial lower bound is known.

In [6] the weighted version of the game was analyzed, in which each player t_i has a weight $w_i \geq 1$ and its cost share of the link is w_i times the link cost, divided by the total weight of the players using the link. They proved that in directed networks a pure-strategy Nash equilibrium does not always exist, and gave various results concerning α -approximate Nash equilibria, that is states in which no player can decrease its payment by more than an α multiplicative factor.

In [5] the authors considered undirected networks and proved that finding a Nash equilibrium that minimizes the potential function is NP-hard. They also focused on the price of anarchy of Nash equilibria resulting from best-response walks from the empty initial state, that is with players joining the game sequentially. For a game with r players, they established an upper bound of $O(\sqrt{r} \log^2 r)$ on the price of anarchy, and a lower bound of $\Omega(\log r / \log \log r)$. Such an equilibrium is reached after a 1-round walk and thus provides an $\Omega(\log r / \log \log r)$ lower bound on the price of anarchy reached after any k -round walk with $k \geq 1$. For 1-round walks, a slightly better upper bound of $O(\sqrt{r} \log r)$ was proven, while an increased $\Omega(\log r)$ lower bound comes directly from a construction given in [21], even if the achieved path system after the walk is not an equilibrium.

Finally, in [14] different reasonable cost sharing methods in undirected networks were considered including the Shapley and egalitarian ones, and their performances have been investigated versus two possible global criteria: the overall cost of the used links and the maximum shared cost of the receivers. Among the various results, the methods achieving a price of anarchy comparable to the one at equilibrium already after one round have been determined by providing corresponding upper and lower bounds after 1-round walks. In particular, a price of anarchy of $\Theta(r^2)$ has been proved for 1-round walks under the Shapley method, that can be directly extended also to directed networks.

Other results explicitly taking into account the speed of convergence of the price of anarchy after a fixed number of rounds concerned basic valid-utility games [26], potential games [7], load balancing games [11], market sharing games [19], and potential and cut games [7].

To the best of our knowledge, no result explicitly concerned the determination of the price of anarchy after any fixed number of rounds $k > 1$, and for any $k \geq 1$ when the underlying communication network is directed, even if for $k = 1$ the undirected results in [14] can be extended directly.

Our Contribution.

In this paper we first consider the multicast game in directed networks induced by the *Shapley Value* cost sharing method [33], which equally distributes the cost of each link among all the down-stream receivers.

More precisely, we evaluate the price of anarchy after a limited number of best response moves, that is after k -round walks for fixed values of $k \geq 1$. Namely, assuming that the initial set of connecting paths can be arbitrary, we show an $O(r\sqrt{r})$ upper bound for 2-round walks, and a matching lower bound that we also extend to $\Omega(r \sqrt[k]{r})$ for k -round walks with $k \geq 2$. Similarly, exactly matching upper and lower bounds equal to r are determined for any number of

rounds when starting from the empty state in which no communication path has been selected.

We then prove similar results also for three other natural cost sharing methods proposed in the literature, which distribute the cost as follows: (i) in an *egalitarian* way, that is by equally distributing the overall cost among all the receivers; (ii) in a *path-proportional* way, that is by distributing the cost of each link among its down-streaming receivers proportionally to the overall cost their chosen path requires; (iii) in an *egalitarian-path-proportional* way, that is by distributing the overall transmission cost among all the receivers proportionally to the cost of their chosen path.

Finally, many results for all the four cost sharing methods are also extended to the undirected case in which the communication links are bidirectional.

The paper is organized as follows. In the next section we present some basic definitions and notation. In Section 3 we present our results on the Shapley cost sharing method and in Section 4 on the three remaining ones. In Section 5 we extend our results to undirected networks and finally, in Section 6, we give some conclusive remarks and discuss some open questions.

Due to space limitations, all the figures are in the appendix.

2. MODEL

We model our communication network as a directed graph $G(V, E, c)$ in which $V = \{v_1, v_2, \dots, v_n\}$ is a set of intercommunicating nodes, $E \subseteq V \times V$ is a set of m links between the nodes and $c : E \mapsto \mathbb{R}^+$ is a function associating to each link (v_i, v_j) a *transmission cost*, that is the cost for exchanging messages between nodes v_i and v_j . Given a subset of links $F \subseteq E$, we denote by $c(F)$ the sum of the costs of the links in F , i.e., $c(F) = \sum_{e \in F} c(e)$. For the sake of simplicity we identify any path p in G with the set of its traversed links, with $c(p) = \sum_{e \in p} c(e)$ being its cost.

We consider the scenario where there is a special node $s \in V$ called *source*, and a set of r nodes $R = \{t_1, t_2, \dots, t_r\} \subseteq V - \{s\}$ called *receivers* (also called *terminals* or *players*) representing the set of users interested in receiving the transmission from the source s . The goal of each receiver is to choose the path from the source that, given the choices of the other receivers, minimizes its payment, determined according to a particular cost sharing method.

More formally, we refer to the set of all the possible paths from the source to t_i as the *strategy set* of a receiver t_i , denoted by P_i , and to the path $p_i \in P_i$ chosen by t_i as the *strategy* of t_i . We denote by m_i a path of minimum cost in P_i , that is such that $c(m_i) = \min_{p \in P_i} c(p)$, and with m^* a path of maximum minimum cost, i.e., having $c(m^*) = \max_{t_i \in R} c(m_i)$. At any time the combination of all the receivers' strategies yields a *path system* $\mathcal{P} = (p_1, p_2, \dots, p_r) \in \mathcal{S}$, where \mathcal{S} is the set of all the possible path systems, i.e., $\mathcal{S} = P_1 \times P_2 \times \dots \times P_r$. We let $\mathcal{P} \oplus p'_i = (p_1, \dots, p_{i-1}, p'_i, p_{i+1}, \dots, p_r)$, that is, the path system obtained from \mathcal{P} if receiver i changes its strategy from p_i to p'_i , and denote by $cost(\mathcal{P})$ the overall transmission cost of the path system \mathcal{P} , which is obtained by summing up the cost of all the links belonging to \mathcal{P} , i.e., $cost(\mathcal{P}) = c(F)$, where $F = \bigcup_{i=1}^r p_i$.

Given a path system \mathcal{P} , the payment attributed to t_i is determined according to a given *cost sharing method*, that

is a function \mathcal{M} which distributes among all the receivers the total cost $cost(\mathcal{P})$ in such a way that $\sum_{t_i \in R} \mathcal{M}(\mathcal{P}, i) = cost(\mathcal{P})$, where $\mathcal{M}(\mathcal{P}, i)$ is the payment of the receiver t_i .

We consider the following four natural cost sharing methods:

- \mathcal{M}_1 (*Shapley* [33]) equally distributes the cost of each link among all the receivers using it, i.e., $\mathcal{M}_1(\mathcal{P}, i) = \sum_{e \in p_i} \mathcal{M}_1^e(\mathcal{P}, i)$, where $\mathcal{M}_1^e(\mathcal{P}, i)$ is the payment of terminal t_i for the link e , defined as $\mathcal{M}_1^e(\mathcal{P}, i) = \frac{c(e)}{n_e(\mathcal{P})}$, with $n_e(\mathcal{P}) = |\{t_i \in R \mid e \in p_i\}|$ being the number of receivers using link e for their transmission.
- \mathcal{M}_2 (*egalitarian* [10]) equally distributes the overall cost $cost(\mathcal{P})$ among all the receivers, i.e., $\mathcal{M}_2(\mathcal{P}, i) = \frac{cost(\mathcal{P})}{|R|}$.
- \mathcal{M}_3 (*path-proportional*) distributes the cost of each link among all the receivers using it proportionally to the cost of their chosen path, i.e., $\mathcal{M}_3(\mathcal{P}, i) = \sum_{e \in p_i} \mathcal{M}_3^e(\mathcal{P}, i)$, where $\mathcal{M}_3^e(\mathcal{P}, i)$ is the payment of terminal t_i for the link e , defined as $\mathcal{M}_3^e(\mathcal{P}, i) = c(e) \frac{c(p_i)}{\sum_{i' : e \in p_{i'}} c(p_{i'})}$.
- \mathcal{M}_4 (*egalitarian-path-proportional*) distributes the overall cost $cost(\mathcal{P})$ among all the receivers proportionally to the cost of their chosen path, i.e., $\mathcal{M}_4(\mathcal{P}, i) = cost(\mathcal{P}) \frac{c(p_i)}{\sum_{t_{i'} \in R} c(p_{i'})}$.

A *Nash equilibrium* is a path system such that no receiver can reduce its payment by changing its strategy given the strategies of the other receivers. More formally, a Nash equilibrium is a path system $\mathcal{P} = (p_1, p_2, \dots, p_r)$ such that $\forall t_i \in R$ and path $p'_i \in P_i$, it holds $\mathcal{M}(\mathcal{P}, i) \leq \mathcal{M}(\mathcal{P} \oplus p'_i, i)$. Denoting with \mathcal{N} the set of all the possible Nash equilibria, the *price of anarchy* (*PoA*) is defined as the worst case ratio among the Nash versus optimal performance, i.e., $PoA(G, R, \mathcal{M}) = \frac{\max_{\mathcal{P} \in \mathcal{N}} cost(\mathcal{P})}{OPT(G, R)}$, where $OPT(G, R) = \min_{\mathcal{P} \in \mathcal{S}} cost(\mathcal{P})$.

In the following, when clear from the context, we will denote $PoA(G, R, \mathcal{M})$ and $OPT(G, R)$ simply as *PoA* and *OPT*, respectively.

We assume that the acting receiver always chooses the strategy that minimizes its payment, given the strategy of the other receivers, or in other words that the strategy of the current receiver is a *best response move* to the other receivers' strategies. When the receiver cannot (strictly) decrease its payment seceding in favor of better path, it is assumed that it plays a best response move consisting in maintaining its current strategy.

In order to model the selfish behavior of the receivers, let us introduce the notion of *state graph*.

Definition 1. A *state graph* $\mathcal{G}(\mathcal{M})$ is a directed graph having a node for any possible path system $\mathcal{P} = (p_1, p_2, \dots, p_r)$ (also called *state*) and an arc $(\mathcal{P}, \mathcal{P}')$ with label i , where $\mathcal{P}' = \mathcal{P} \oplus p'_i$ and $p'_i \in P_i$, iff both these conditions are met: (i) $\mathcal{M}(\mathcal{P}', i) \leq \mathcal{M}(\mathcal{P} \oplus p''_i, i)$ for every $p''_i \in P_i$; (ii) if $\mathcal{P} \neq \mathcal{P}'$, $\mathcal{M}(\mathcal{P}', i) < \mathcal{M}(\mathcal{P}, i)$.

Notice that the graph may contain loops and there is an arc $(\mathcal{P}, \mathcal{P}')$ labeled i if and only if t_i , starting from \mathcal{P} , can play a best response move resulting in the path system \mathcal{P}' .

Any path in the state graph is called a *best response walk*. Given a best response walk starting from an arbitrary state, we are interested in the cost of the last state of the walk. Notice that if we do not allow every player to make a best response on a walk, then we cannot bound the cost of the final state with respect to the optimal solution. This follows from the fact that the actions of a single player may be very important for producing solutions of high social value. Motivated by this simple observation, the following definitions capture the intuitive notion of a fair sequence of moves [26]:

1-round walk: given an arbitrary ordering of all receivers t_1, t_2, \dots, t_r , it is a walk of length r in the state graph in which the arcs are labelled in the order $1, 2, \dots, r$.

k-round walk: a walk in the state graph that can be split in k disjoint 1-round walks.

Extending the classical definition, we let $PoA_k(G, R, \mathcal{M})$ or simply PoA_k be the price of anarchy yielded by k -round walks, that is the worst case ratio among a path system corresponding to the last state of a k -round walk and OPT . We will consider the following two scenarios:

1. all walks are assumed to start from an arbitrary initial state;
2. all walks are assumed to start from the “empty” state in which no receiver has still selected its communication path from the source; in order to include such a situation in the above framework we include an additional special empty path \emptyset in the strategies sets P_i , assuming that the only possible best response moves from \emptyset are the ones selecting a path of minimum payment actually connecting the source to the receiver; all the notation and definitions are trivially extended accordingly; in order to distinguish from the previous case, the price of anarchy will be denoted as $PoA - Empty_k(G, R, \mathcal{M})$ or simply $PoA - Empty_k$.

3. RESULTS ON THE SHAPLEY METHOD

In this section we show our results concerning the Shapley cost sharing method \mathcal{M}_1 , which is the fundamental and mostly considered one in the literature under this setting. More precisely, we provide an upper for 2-round walk and a matching lower bound that we extend also to k -round walks, assuming that walks can start from arbitrary initial state. We then provide exactly matching bounds on the price of anarchy also when starting from the empty configuration.

Let us first consider the case of arbitrary initial states. We consider only k -round walks for $k \geq 2$, as for $k = 1$ matching lower and upper bounds proportional to r^2 can be inferred directly from the results in [14].

The following lemma will be useful in the sequel.

LEMMA 1. *Given any two sequences of $k > 0$ positive real numbers a_1, a_2, \dots, a_k and b_1, b_2, \dots, b_k , the following inequality holds:*

$$\sum_{i=1}^k \frac{a_i}{b_i} \geq \frac{\left(\sum_{i=1}^k a_i\right)^2}{\sum_{i=1}^k a_i b_i}.$$

PROOF. The lemma follows by observing that

$$\begin{aligned} \sum_{i=1}^k \frac{a_i}{b_i} \sum_{i=1}^k a_i b_i &= \sum_{i=1}^k a_i^2 + \sum_{i=0}^k \sum_{j>i}^k a_i a_j \left(\frac{b_i}{b_j} + \frac{b_j}{b_i}\right) \geq \\ &\geq \sum_{i=1}^k a_i^2 + 2 \sum_{i=0}^k \sum_{j>i}^k a_i a_j = \left(\sum_{i=1}^k a_i\right)^2, \end{aligned}$$

since $\left(\frac{a}{b} + \frac{b}{a}\right) \geq 2$ for any pair of positive real numbers a, b . \square

The following lemma states a general property of the path systems obtained during a 1-round walk starting from an arbitrary state. Informally, assuming the receivers listed in order of move, that is that t_i moves at the i -th step, denoted as \tilde{p}_i^r the set of new edges added by t_i to the ones used by t_1, \dots, t_{i-1} , if the cost of \tilde{p}_i^r is high, then its edges must be shared by many of the not yet moving receivers.

LEMMA 2. *Let \mathcal{W} be a 1-round walk in $\mathcal{G}(\mathcal{M}_1)$ starting from an arbitrary state $\mathcal{P}^0 = (p_1^0, p_2^0, \dots, p_r^0)$, and let p_i^r be the path chosen by the receiver t_i in the intermediate state $\mathcal{P}^{i-1} = (p_1^r, p_2^r, \dots, p_{i-1}^r, p_i^0, \dots, p_r^0)$ of \mathcal{W} , resulting in the state $\mathcal{P}^i = \mathcal{P}^{i-1} \oplus p_i^r$. Then $\sum_{j>i} c(p_j^0 \cap \tilde{p}_i^r) \geq (\alpha_i - 1)\alpha_i c(m^*)$, where $\tilde{p}_i^r = \bigcup_{j=1}^i p_j^r \setminus \bigcup_{j=1}^{i-1} p_j^r$ and $\alpha_i = \frac{c(\tilde{p}_i^r)}{c(m^*)}$.*

PROOF. Since the acting receiver t_i performs a best response move, it chooses a path p_i^r such that its payment is less or equal than $c(m^*)$, i.e., $c(m^*) \geq \mathcal{M}_1(\mathcal{P}^i, i)$. By the definition of \mathcal{M}_1 and since $\tilde{p}_i^r \subseteq p_i^r$, we get

$$c(m^*) \geq \sum_{e \in \tilde{p}_i^r} \frac{c(e)}{n_e(\mathcal{P}^i)} \geq \sum_{e \in \tilde{p}_i^r} \frac{c(e)}{n_e(\mathcal{P}^i)},$$

and by the definition of α_i ,

$$c(\tilde{p}_i^r) \geq \alpha_i \sum_{e \in \tilde{p}_i^r} \frac{c(e)}{n_e(\mathcal{P}^i)}.$$

By Lemma 1, $\sum_{e \in \tilde{p}_i^r} \frac{c(e)}{n_e(\mathcal{P}^i)} \geq \frac{c(\tilde{p}_i^r)^2}{\sum_{e \in \tilde{p}_i^r} c(e)n_e(\mathcal{P}^i)}$, so that by exploiting the previous inequality,

$$\sum_{e \in \tilde{p}_i^r} c(e)n_e(\mathcal{P}^i) \geq \alpha_i c(\tilde{p}_i^r).$$

Clearly, since every edge $e \in \tilde{p}_i^r$ is included in none of the paths p_1^r, \dots, p_{i-1}^r , the quantity $n_e(\mathcal{P}^i)$ is the number of receivers t_j among t_{i+1}, \dots, t_r using e , that is such that $e \in p_j^0$, plus 1 (due to the fact that $e \in \tilde{p}_i^r$). Therefore, $\sum_{e \in \tilde{p}_i^r} c(e)n_e(\mathcal{P}^i) = c(\tilde{p}_i^r) + \sum_{j>i} c(p_j^0 \cap \tilde{p}_i^r)$, so that

$$\sum_{j>i} c(p_j^0 \cap \tilde{p}_i^r) \geq (\alpha_i - 1)c(\tilde{p}_i^r).$$

The lemma follows by observing that $c(\tilde{p}_i^r) = \alpha_i c(m^*)$. \square

Exploiting the above lemma, we now show that, if the initial state of a 1-round walk is not completely arbitrary but for instance yielded by a previous 1-round walk, then the cost of the final path system can be suitably bounded.

LEMMA 3. Let \mathcal{W} be a 1-round walk in $\mathcal{G}(\mathcal{M}_1)$ starting from a state $\mathcal{P}^0 = (p_1^0, p_2^0, \dots, p_r^0)$ such that $c(p_i^0) \leq rc(m^*)$ for every receiver t_i . Moreover, let $\mathcal{P}^r = (p_1^r, p_2^r, \dots, p_r^r)$ be the last state in \mathcal{W} . Then $\frac{\text{cost}(\mathcal{P}^r)}{\text{OPT}} = O(r\sqrt{r})$.

PROOF. For every i , $1 \leq i \leq r$, assuming that receivers move according to the order t_1, \dots, t_r , let $\tilde{p}_i^r = \bigcup_{j=1}^i p_j^r \setminus \bigcup_{j=1}^{i-1} p_j^r$ and $\alpha_i = \frac{c(\tilde{p}_i^r)}{c(m^*)}$. Then, since $\tilde{p}_1^r, \dots, \tilde{p}_r^r$ partition the edges of \mathcal{P}^r ,

$$\text{cost}(\mathcal{P}^r) = \sum_{i=1}^r c(\tilde{p}_i^r) = c(m^*) \sum_{i=1}^r \alpha_i. \quad (1)$$

Moreover, as it can be immediately checked $\sum_{i=1}^r c(p_i^0) \geq \sum_{i=1}^r \sum_{j>i} c(p_j^0 \cap \tilde{p}_i^r)$, and since $rc(m^*) \geq c(p_i^0)$ for every receiver t_i , $r^2 c(m^*) \geq \sum_{i=1}^r c(p_i^0)$. Therefore, by Lemma 2, it follows that

$$r^2 \geq \sum_{i=1}^r (\alpha_i^2 - \alpha_i). \quad (2)$$

Since $\sum_{i=1}^k a_i^2 \geq \frac{1}{k} \left(\sum_{i=1}^k a_i \right)^2$ for any sequence of $k > 0$ real numbers a_1, a_2, \dots, a_k , by exploiting inequality 2, $r^2 \geq \frac{1}{r} \left(\sum_{i=1}^r \alpha_i \right)^2 - \sum_{i=1}^r \alpha_i$, from which it immediately follows that

$$\sum_{i=1}^r \alpha_i \leq r \left(\sqrt{(r+1/4)} + 1/2 \right). \quad (3)$$

The lemma then follows from the inequalities 1 and 3 by observing that $c(m^*) \leq \text{OPT}$. \square

As a direct consequence of Lemma 3 the following theorem holds.

THEOREM 1. $PoA_2 = O(r\sqrt{r})$.

PROOF. The claim follows directly from Lemma 3 by observing that after any 1-round walk the path system $\mathcal{P}^0 = (p_1^0, p_2^0, \dots, p_r^0)$ from which the second 1-round walk starts is such that $c(p_i^0) \leq rc(m^*)$ for every receiver t_i . In fact, p_i^0 has been chosen by t_i during the first 1-round walk and every path longer than $rc(m^*)$ would result in a payment strictly greater than $c(m^*)$, regardless of the particular state. Therefore, it is never chosen by a receiver, as it can always select the strategy consisting of the minimum cost connection, which requires a payment at most equal to $c(m^*)$. \square

We now prove a matching lower bound.

THEOREM 2. $PoA_2 = \Omega(r\sqrt{r})$.

PROOF. We build a network with an arbitrarily high number of receivers r in which there exists a 2-round walk leading to a state of social cost $\Omega(r\sqrt{r})$, while $\text{OPT} = O(1)$.

Assuming that $r = j^2$ for some integer $j > 0$ and that ϵ is a suitably small number such that $\epsilon \leq \left(1 - \frac{j}{\sqrt{r}}\right)$, the network is constructed as follows (see Fig. 1). The source node s is connected to one node u_1 with a link of cost ϵ and to $\sqrt{r} - 1$ other nodes $u_2, \dots, u_{\sqrt{r}}$ with links of costs 1. Such nodes are said of the *first level*. Each u_i , $1 \leq i \leq \sqrt{r}$, is

connected to one node $u_{i,1}$ with a link of cost ϵ and to $\sqrt{r} - 1$ other nodes $u_{i,2}, \dots, u_{i,\sqrt{r}}$ with links of costs 2. Such nodes are said of the *second level*. Every $u_{i,j}$ has a link of cost $\sqrt{r} + 1 - j$ toward receiver $t_{(i-1)\sqrt{r}+j}$. In addition, arcs of cost ϵ connect every receiver $t_{(i-1)\sqrt{r}+j}$ with $j < \sqrt{r}$ to node $u_{i,j+1}$ and every $t_{i\sqrt{r}}$ with $i < \sqrt{r}$ to $u_{i+1,1}$. Finally, we add two other nodes q_1 and q_2 connected in such a way that q_1 has an incoming link of cost 4 from s and an outgoing link of cost ϵ to each receiver, while q_2 has an incoming link of cost ϵ from s and an outgoing link of cost 4 to each receiver (again see Fig. 1, where arcs without explicit costs are weighted ϵ).

It's easy to see that the optimal solution is the path system obtained when each receiver chooses the unique path connecting s to it through node q_1 . Hence we obtain $\text{OPT} = 4 + r\epsilon$.

Let us analyze the evolution of the game starting from the configuration in which all the receivers $t_{(i-1)\sqrt{r}+j}$ are reached from the source by means of the unique path going through $u_{1,1}$ (and passing through all the previous receivers $t_{(i'-1)\sqrt{r}+j'}$ with $i' \leq i$ and $j' \leq j$). In this configuration the links used for the transmission are (s, u_1) , $(u_1, u_{1,1})$ and the ones between the second level and the receivers.

We assume that the receivers play in the order t_1, t_2, \dots, t_r during both the two rounds. Each receiver $t_{(i-1)\sqrt{r}+j}$ can select a path from 3 possible sets:

- S_i^1 containing all the paths going through $u_{i,1}$;
- S_i^2 containing all the paths going through some node $u_{i,j}$ with $2 \leq j \leq \sqrt{r}$ and not through $u_{i,1}$;
- S_i^3 containing the paths going through q_1 or q_2 .

Notice that the set S_i^2 is empty for receiver $t_{(i-1)\sqrt{r}+1}$, which is adjacent to $u_{i,1}$.

Consider the point during the first 1-round walk in which all the receivers $t_{(i-1)\sqrt{r}+j}$ must do a best-response move, and assume that if $i > 1$ all the receivers $t_{(i'-1)\sqrt{r}+j'}$ with $i' < i$ have selected a communication path in $S_{i'}^1$.

Since all the receivers $t_{(i'-1)\sqrt{r}+j'}$ with $i' > i$ have not moved yet and thus are using the initial path from $u_{1,1}$, the path $\langle s, u_i, u_{i,1}, t_{(i-1)\sqrt{r}+1}, u_{i,2}, t_{(i-1)\sqrt{r}+2}, \dots, u_{i,j}, t_{(i-1)\sqrt{r}+j} \rangle$ for receiver $t_{(i-1)\sqrt{r}+j}$ whose subpath $\langle u_{i,1}, t_{(i-1)\sqrt{r}+1}, u_{i,2}, t_{(i-1)\sqrt{r}+2}, \dots, u_{i,j}, t_{(i-1)\sqrt{r}+j} \rangle$ of cost $\left(\frac{\sqrt{r}(\sqrt{r}+1)}{2} + \epsilon(\sqrt{r}-1) \right)$ is used by all the $\sqrt{r}(\sqrt{r}-i)$ receivers $t_{(i'-1)\sqrt{r}+j'}$ with $i' > i$ requires a payment less than

$$(1 + \epsilon) + \left(\frac{\sqrt{r}(\sqrt{r}+1)}{2} + \epsilon(\sqrt{r}-1) \right) \frac{1}{\sqrt{r}(\sqrt{r}-i)} <$$

$$< \frac{\sqrt{r}(\sqrt{r}+1)}{2(\sqrt{r}-i)} + 2 < 3$$

$$\text{as } \epsilon \leq \left(1 - \frac{j}{\sqrt{r}}\right) \text{ and } i \leq \frac{\sqrt{r}-1}{2}.$$

Since all the paths in S_i^2 and S_i^3 require a payment at least equal to 3, $t_{(i-1)\sqrt{r}+j}$ and thus all the $t_{(i-1)\sqrt{r}+j'}$ with $1 \leq j' \leq \sqrt{r}$ choose a path in S_i^1 .

Therefore, at the end of the 1-round walk all the receivers $t_{(i-1)\sqrt{r}+j}$ with $i \leq \frac{\sqrt{r}-1}{2}$ have selected a path in S_i^1 .

Assume that the remaining receivers $t_{(i-1)\sqrt{r}+j}$ with $i > \frac{\sqrt{r}-1}{2}$ never choose a path stepping through q_1 (the first among them such that its best response moves include the two equivalent paths stepping through q_1 and q_2 , respectively, will choose q_2 , and so the successive ones).

During the second 1-round walk, again consider the point in which the receivers $t_{(i-1)\sqrt{r}+j}$ with some fixed $i \leq \frac{\sqrt{r}-1}{2}$ must do a best-response move, and assume that if $i > 1$ all the receivers $t_{(i'-1)\sqrt{r}+j'}$ with $i' < i$ have selected a communication path not in S_i^3 .

Since when a particular $t_{(i-1)\sqrt{r}+j}$ is moving all the $\sqrt{r}-j$ receivers $t_{(i-1)\sqrt{r}+j'}$ with $j' > j$ have not moved yet and thus are using the link $(u_{i,j}, t_{(i-1)\sqrt{r}+j})$ of cost $\sqrt{r} + 1 - j$, the path $\langle s, u_i, u_{i,j}, t_{(i-1)\sqrt{r}+j} \rangle$ requires a payment at most $1 + 2 + \frac{\sqrt{r}+1-j}{\sqrt{r}+1-j} = 4$, while every path in S_i^3 more than 4. Therefore, $t_{(i-1)\sqrt{r}+j}$ does not choose a path in S_i^3 , and thus at the end of the second 1-round walk all the receivers $t_{(i-1)\sqrt{r}+j}$ with $i \leq \frac{\sqrt{r}-1}{2}$ have selected a path in S_i^1 or S_i^2 . Since each such a path contains link $(u_{i,j}, t_{(i-1)\sqrt{r}+j})$, the final path system has cost at least $\frac{\sqrt{r}-1}{2} \sum_{j=1}^{\sqrt{r}} (\sqrt{r} + 1 - j) = \frac{\sqrt{r}-1}{2} \frac{\sqrt{r}(\sqrt{r}+1)}{2} = \frac{(r-1)\sqrt{r}}{4}$, hence the theorem. \square

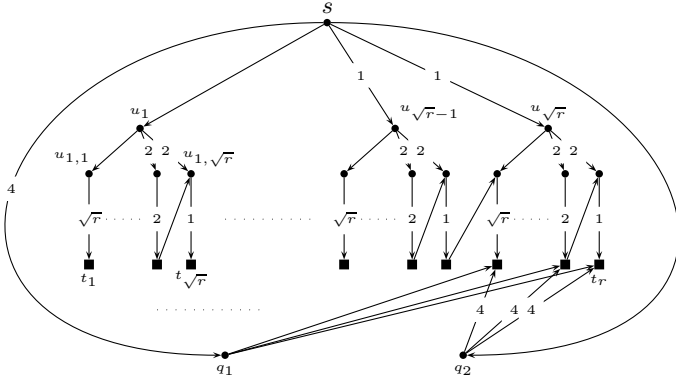


Figure 1: The directed lower bound network for 2-round walks from arbitrary state under the Shapley method.

The above lower bound can be extended to any number $k \geq 2$ of rounds.

THEOREM 3. $PoA_k = \Omega(r \sqrt[k]{r})$ for every $k \geq 2$.

Let us now focus on walks starting from the empty state. Under such an assumption, the following theorem holds.

THEOREM 4. $PoA - Empty_k = r$ for every $k \geq 1$.

PROOF. Let us first show that $PoA - Empty_k \geq r$. To this aim, consider the network depicted in Figure 2, where ϵ is a suitably small number. Starting from the empty state, every receiver t_i , $1 \leq i \leq r$, chooses the path consisting of the unique link connecting it to the source. Thus at the end of the 1-round walk the final path system will cost r , while the optimal solution is given by all the paths $\langle s, q, t_i \rangle$, that yield an overall cost equal to $1 + \epsilon r$. The lower bound thus follows from the arbitrariness of ϵ .

In order to show that $PoA - Empty_k \leq r$, consider the classical potential function $\Phi(\mathcal{P}) = \sum_{e \in E} \sum_{i=1}^{n_e(\mathcal{P})} \frac{c(e)}{i}$ associated to every path system \mathcal{P} . Then, since during the first round every moving receiver t_i chooses a strategy of payment at most $c(m_i) \leq OPT$ (the cost of a minimum connecting path), that is it increases the potential function at most of OPT , if \mathcal{P} is the path system achieved at the end of the round, it results $\Phi(\mathcal{P}) \leq r$. The theorem follows by observing that the potential function can only decrease at every best response move after the first round, so that every path system \mathcal{P}' reached after \mathcal{P} is such that $cost(\mathcal{P}') \leq \Phi(\mathcal{P}') \leq \Phi(\mathcal{P}) \leq r$. \square

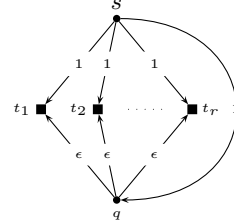


Figure 2: The directed lower bound network from the empty state for Shapley and the other methods.

4. RESULTS ON THE OTHER COST SHARING METHODS

We now focus on the remaining cost sharing methods.

Let us first consider the case in which walks can start from an arbitrary initial state.

For the egalitarian cost sharing method we first observe that the price of anarchy is unbounded after any fixed number $k \geq 1$ steps. In fact, for every h suitably large, there exists a Nash equilibrium having price of anarchy at least equal to h [14].

Unfortunately, the same negative result holds also for the path-proportional method.

THEOREM 5. $PoA_k(G, R, \mathcal{M}_3) \geq h$ for every $h > 0$.

PROOF. For every fixed k , we show that there exists a network in which the cost of the path system obtained after a k -round walk starting from an arbitrary initial state in $\mathcal{G}(\mathcal{M}_3)$ is greater than h for every $h > 0$.

Consider the network depicted in Figure 3 with $r = k + 1$ receivers, where M is a suitably large number. Assuming that the initial communication path of each t_i , $1 \leq i \leq r$, is $\langle s, t_1, \dots, t_i \rangle$ and that in every 1-round walk receivers move according to the order t_1, t_2, \dots, t_r , during the first 1-round walk t_1, \dots, t_{r-1} maintain the currently selected paths. In fact, the payment of each receiver t_i , $1 \leq i \leq r - 1$, is

at most $\frac{(\sum_{j=1}^i M^{2^j-1})^2}{(\sum_{j=1}^i M^{2^j-1}) + M^{2^{i+1}-1}} = \Theta(\frac{1}{M})$ (the value that t_i would pay for his current path if shared only with t_{i+1}), which, for M suitably large, is strictly less than the one

required by the only possible alternative path consisting of the single link (s, t_i) . Only t_r selects the single link path (s, t_r) of cost 1. In fact, in the initial communication path, it would pay for the total cost of the link (t_r, t_{r-1}) .

Continuing in this fashion, by completely analogous considerations, it is easy to see that during the i -th 1-round walk, t_1, \dots, t_{r-i} maintain their current path, t_{r-i+1} selects the link (s, t_{r-i+1}) , and t_{r-i+2}, \dots, t_r maintain their single link path from the source.

Therefore, at the end of the k -th 1-round walk all the receivers but t_1 have changed the initial communication path to the single link from the root of cost 1, while t_1 still maintains the initial link (s, t_1) of cost M , resulting in a path system of total cost $r - 1 + M$. The optimal path system is obtained by selecting the link (s, t_1) of cost 1 also for t_1 and has cost r . The theorem thus follows for M suitably large. \square

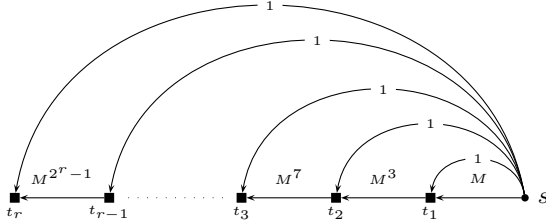


Figure 3: The directed path-proportional lower bound network from arbitrary state.

Finally, for the egalitarian-path-proportional method, in [14] it has been shown that the price of anarchy is $O(r)$ after every 1-round walk starting from an arbitrary initial state, so the same holds after any k -round walk with $k \geq 1$. Such a result is asymptotically optimal, as there exist Nash equilibria with price of anarchy r .

For all the above three cost sharing methods, the following theorem holds for walks starting from the empty state.

THEOREM 6. $PoA - Empty_k(G, R, \mathcal{M}_j) = \Theta(r)$ for every $k \geq 1, 2 \leq j \leq 4$.

PROOF. The lower bound proof exploits the same lower bound network of Theorem 4 and is completely analogous.

In order to show that $PoA - Empty_k(G, R, \mathcal{M}_2) = O(r)$ and $PoA - Empty_k(G, R, \mathcal{M}_3) = O(r)$, it suffices to observe that at the every round the moving receivers can increase the cost of the initial path system at most of $rOPT$. In fact, every time a receiver t_i moves, the set of new edges it adds to the path system cannot cost more than $c(m_i) \leq OPT$ (the cost of a minimum connecting path), thus giving in total an additional cost of at most $rOPT$ for every single round. For the egalitarian method a refined exactly matching upper bound $PoA - Empty_k(G, R, \mathcal{M}_2) \leq r$ directly follows by observing that $rOPT$ is an upper bound on the cost of the path system achieved after the first round and that the successive best response moves cannot increase the cost of the induced path system.

Finally, for the egalitarian-path-proportional method, as already mentioned above, from [14] it follows that the price of anarchy is $O(r)$ after any k -round walk with $k \geq 1$ even starting from an initial arbitrary state. \square

5. THE UNDIRECTED CASE

In this section we briefly discuss the extension of our result to the case in which the underlying communication network is undirected.

If we not claim it explicitly, all our results from arbitrary initial state can be extended to the undirected case, with the exception of the lower bounds for the Shapley method stated in Theorems 2 and 3. However, in this case $PoA_k(G, R, \mathcal{M}_1) \geq r$ for every $k \geq 1$ directly follows by observing that there exists a Nash equilibrium with price of anarchy r .

On the contrary, many differences hold when starting from empty state. In fact, in this case the determination of the price of anarchy for the Shapley method is an important open question with a significant gap between the known lower and the upper bound. More precisely, a $\Omega(\log r / \log \log r)$ lower bound for any number of rounds comes directly from the results in [5], where an equilibrium with such a price of anarchy is reached in one round. In [5] also an upper bound for any number of rounds of $O(\sqrt{r} \log^2 r)$ has been proven. For 1-round walks, a slightly better upper bound of $O(\sqrt{r} \log r)$ again has been shown in [5], while an increased $\Omega(\log r)$ lower bound comes directly from a construction given in [21].

Concerning the egalitarian method, $PoA - Empty_1(G, R, \mathcal{M}_2) = \Theta(\log r)$ derives directly by observing that a 1-round walk starting from the empty state can be seen as the execution of the greedy online Steiner tree algorithm of [21], which has competitive ratio $\Theta(\log r)$.

The path-proportional method seems also very difficult, as it can be seen as a refinement of the Shapley one. In particular, while again a $\Omega(\log r)$ lower bound for 1-round walks can be derived from the construction provided in [21], we can prove as in the directed case that $PoA - Empty_k(G, R, \mathcal{M}_3) = O(r)$.

Finally, for the egalitarian-path-proportional method, while again $PoA - Empty_k(G, R, \mathcal{M}_4) = O(r)$ for every $k \geq 1$ can be proven as in directed case, the following lower bound holds.

THEOREM 7. $PoA - Empty_1(G, R, \mathcal{M}_4) = \Omega(r / \log r)$.

PROOF. We show that there exists a network in which the cost of the path system obtained after a 1-round walk starting from the empty state is $\Omega(r)$, while $OPT = O(\log r)$.

To this aim consider the network depicted in Fig. 4 and assume that receivers move according to the order t_1, \dots, t_r . Moreover in such a network, since all the communication paths for receiver t_1 require the same payment equal to 1, assume that t_1 chooses the communication path consisting of the single link (s, t_1) of cost 1.

Then, receiver t_2 selects the path consisting of the single link (s, t_2) of cost $1/2$. In fact, this requires payment $1/2$, while the cheapest alternative possible path, that is the one stepping through t_1 , has payment $9/10$.

In general, assuming that all the previous receivers have chosen as communication path the single link from the source, also $t_i, 2 < i \leq r$, would choose link (s, t_i) . In fact, denoted as $C_{i-1} = 1 + \frac{1}{2} + \dots + (1 - \frac{1}{i-1})$ the total cost of the links chosen by receivers t_1, \dots, t_{i-1} , it requires a payment equal to $\frac{(C_{i-1} + 1 - \frac{1}{i})(1 - \frac{1}{i})}{(C_{i-1} + 1 - \frac{1}{i})} = 1 - \frac{1}{i}$, while the cheapest alternative path, that is the one stepping through

t_1 , has payment equal to $\frac{(C_{i-1} + \frac{1}{i})(1 + \frac{1}{i})}{(C_{i-1} + 1 + \frac{1}{i})}$, that as it can be easily checked is strictly greater than $1 - \frac{1}{i}$ for every $i \geq 2$.

The theorem then follows by observing that the final path system will cost $C_r = 1 + \frac{1}{2} + \dots + (1 - \frac{1}{r}) \geq r - \ln(r) - 1$, while the optimal path system would be induced by the communication paths (s, t_1) for t_1 and (s, t_1, t_i) for every t_i , $2 \leq i \leq r$, whose total cost is $1 + \frac{1}{2} + \dots + \frac{1}{r} \leq \ln r + 1$. \square

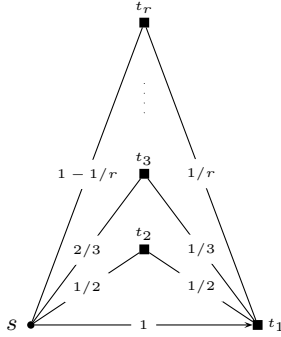


Figure 4: The undirected egalitarian-path-proportional lower bound network from the empty state.

6. CONCLUSION AND OPEN QUESTIONS

We have investigated the price of anarchy after a limited number of steps reached according to four natural cost sharing methods.

The rational of our investigation is that, starting from arbitrary initial states, only the egalitarian-path-proportional method is able to reach a price of anarchy comparable to the one at equilibria after a 1-round walk, while for the remaining methods either this is not possible even after k -round walks for any fixed $k \geq 1$ or the price of anarchy is unbounded also at equilibrium.

Particularly interesting is the case in which walks start from the empty state. In fact, the cost sharing methods not having Nash equilibria or bounded price of anarchy at equilibrium reach a price of anarchy proportional to r after a single 1-round walk.

Many questions are left open.

First of all, there is the reduction of the gap between the lower and upper bounds in many considered cases. In this setting, for the Shapley method particularly relevant would be the determination of matching bounds in the directed case for $k > 2$ rounds. While the $O(r\sqrt{r})$ upper bound holds also for every $k > 2$, we conjecture that a better bound holds, but its determination appears untrivial. In fact, the proof should deeply exploit the combinatorial structure of the path systems reached after two rounds, as the constraint on the cost of the connecting paths and of their induced path system are not sufficient. This stems from the fact that even at Nash equilibrium such paths can cost r times the optimum social cost, while exploiting the fact that the path system after the second round has a cost proportional to $r\sqrt{r}$ times the optimum cannot lead to better results. In fact, in

our $O(r\sqrt{r})$ lower bound on the price of anarchy after two rounds, the second round started from a path system already of cost proportional to $r\sqrt{r}$ times the optimum, so in such a case the second round has not asymptotically decreased the path system cost. In the undirected case the closure of the $\Omega(\log r) \div O(\sqrt{r} \log r)$ gap between the lower and the upper bound after one round starting from the empty state is one of the purposes of many researchers.

Moreover, with respect to the other cost sharing methods, in the undirected case it would be worth to determine the exact price of anarchy after $k > 1$ rounds starting from the empty state for the egalitarian method and after $k \geq 1$ rounds from empty state for the other ones.

Finally, another worth investigating issue would be that of bounding the price of stability after any limited number of rounds.

7. REFERENCES

- [1] E. Anshelevich, A. Dasgupta, J. M. Kleinberg, E. Tardos, T. W., and T. Roughgarden. The price of stability for network design with fair cost allocation. In *Proceedings of the 45th Symposium on Foundations of Computer Science (FOCS)*, pages 295–304. IEEE Computer Society, 2004.
- [2] A. Archer, J. Feigenbaum, A. Krishnamurthy, R. Sami, and S. Shenker. Approximation and collusion in multicast cost sharing. *Games and Economic Behavior*, 47(1):36–71, 2004.
- [3] V. Bilò, M. Flammini, and L. Moscardelli. On nash equilibria in non-cooperative all-optical networks. In *Proceedings of the 22nd Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, volume 3404 of *LNCS*, pages 448–459. Springer, 2005.
- [4] V. Bilò and L. Moscardelli. The price of anarchy in all-optical networks. In *Proceedings of the 11th Colloquium on Structural Information and Communication Complexity (SIROCCO)*, volume 3104 of *LNCS*, pages 13–22. Springer, 2004.
- [5] C. Chekuri, J. Chuzhoy, L. Lewin-Eytan, J. Naor, and A. Orda. Non-cooperative multicast and facility location games. In *ACM Conference on Electronic Commerce*, pages 72–81. ACM, 2006.
- [6] H. Chen and T. Roughgarden. Network design with weighted players. In *Proceedings of the 18th Annual ACM Symposium on Parallel Algorithms and Architectures (SPAA)*, pages 29–38. ACM, 2006.
- [7] G. Christodoulou, V. S. Mirrokni, and A. Sidiropoulos. Convergence and approximation in potential games. In *Proceedings of the 23rd Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, volume 3884 of *LNCS*, pages 349–360. Springer, 2006.
- [8] V. Conitzer and T. Sandholm. Complexity results about nash equilibria. In *Proceedings of the 18th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 765–771. Morgan Kaufmann, 2003.
- [9] S. Deering and D. Cheriton. Multicast routing in datagram internetworks and extended lans. *ACM Transactions on Computer Systems*, 8:85–110, 1990.
- [10] B. Dutta and D. Ray. A concept of egalitarianism under participation constraints. *Econometrica*, 57:615–635, 1989.

- [11] E. Even-Dar, A. Kesselman, and Y. Mansour. Convergence time to nash equilibria. In *Proceedings of the 30th International Colloquium on Automata, Languages and Programming (ICALP)*, volume 2719 of *LNCS*, pages 502–513. Springer, 2003.
- [12] A. Fabrikant, A. Luthra, E. Maneva, C. H. Papadimitriou, and S. Shenker. On a network creation game. In *Proceedings of the 22nd ACM Symposium on Principles of Distributed Computing (PODC)*, pages 347–351. ACM, 2003.
- [13] A. Fabrikant, C. H. Papadimitriou, and K. Talwar. The complexity of pure equilibria. In *Proceedings of the 36th ACM Symposium on Theory of Computing (STOC)*, pages 604–612. ACM, 2004.
- [14] A. Fanelli, M. Flammini, G. Melideo, and L. Moscardelli. Multicast transmissions in non-cooperative networks with a limited number of selfish moves. In *Proceedings of the 31st International Symposium on Mathematical Foundations of Computer Science (MFCS)*, volume 4162 of *LNCS*, pages 363–374. Springer, 2006.
- [15] J. Feigenbaum, A. Krishnamurthy, R. Sami, and S. Shenker. Hardness results for multicast cost sharing. *Journal of Public Economics*, 304(1-3):215–236, 2003.
- [16] J. Feigenbaum, C. Papadimitriou, and S. Shenker. Sharing the cost of multicast transmissions. In *Proceedings of 32nd ACM Symposium on Theory of Computing (STOC)*, pages 218–227. ACM, 2000.
- [17] D. Fotakis, S. Kontogiannis, E. Koutsoupias, M. Mavronicolas, and P. Spirakis. The structure and complexity of nash equilibria for a selfish routing game. In *Proceedings of the 29th International Colloquium on Automata, Languages and Programming (ICALP)*, LNCS, pages 123–134. Springer, 2002.
- [18] M. X. Goemans, V. S. Mirrokni, and A. Vetta. Sink equilibria and convergence. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 142–154. IEEE Computer Society, 2005.
- [19] M. X. Goemans, E. L. Li, V. S. Mirrokni, and M. Thottan. Market sharing games applied to content distribution in ad-hoc networks. In *Proceedings of the 5th ACM International Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc)*, pages 55–66. ACM, 2004.
- [20] R. Holzman and N. Law-Yone. Strong equilibrium in congestion games. 21:85–101, 1997.
- [21] M. Imaze and B. Waxman. Dynamic steiner tree proble. *SIAM J. on Discrete Mathematic*, 4(3):369–384, 1991.
- [22] K. Jain and V.V. Vazirani. Applications of approximation algorithms to cooperative games. In *Proceedings of the 33rd ACM Symposium on Theory of Computing (STOC)*, pages 364–372. ACM, 2001.
- [23] E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. In *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, volume 1563 of *LNCS*, pages 404–413. Springer, 1999.
- [24] M. Mavronicolas and P. Spirakis. The price of selfish routing. In *Proceedings of the 33rd Annual ACM Symposium on the Theory of Computing (STOC)*, pages 510–519. ACM, 2001.
- [25] I. Milchtaich. Congestion games with player-specific payoff functions. *Games and Economic Behavior*, 13:111–124, 1996.
- [26] V. S. Mirrokni and A. Vetta. Convergence issues in competitive games. In *Proceedings of the 7th International Workshop on Approximation Algorithms for Combinatorial Optimization Problem (APPROX-RANDOM)*, volume 3122 of *LNCS*, pages 183–194. Springer, 2004.
- [27] D. Monderer and L.S. Shapley. Potential games. 14:124–143, 1996.
- [28] J. F. Nash. Equilibrium points in n -person games. In *Proceedings of the National Academy of Sciences*, volume 36, pages 48–49, 1950.
- [29] P. Penna and C. Ventrè. More powerful and simpler cost-sharing methods. In *Proceedings of the 2nd International Workshop on Approximation and Online Algorithms (WAOA)*, volume 3351 of *LNCS*, pages 97–110. Springer, 2004.
- [30] P. Penna and C. Ventrè. Free-riders in steiner tree cost-sharing games. In *Proceedings of the 12th Colloquium on Structural Information and Communication Complexity (SIROCCO)*, volume 3499 of *LNCS*, pages 231–245. Springer, 2005.
- [31] R. W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.
- [32] T. Roughgarden and E. Tardos. How bad is selfish routing? *Journal of ACM*, 49(2):236–259, 2002.
- [33] L.S. Shapley. The value of n -person games. *Contributions to the theory of games*, pages 31–40, Princeton University Press, 1953.
- [34] A. Vetta. Nash equilibria in competitive societies, with applications to facility location, traffic routing and auctions. In *Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 416–425. IEEE Computer Society, 2002.
- [35] M. Voorneveld, P. Borm, F. van Meegen, S. Tijs, and G. Faccini. Congestion games and potentials reconsidered. 1:283–299, 1999.